Dr. Mathias Braun & Wenhao Zhao

Exercises for 'Functional Analysis 2' [MATH-404]

(14/04/2025)

Ex 8.1 (Distributional derivatives as difference quotients)
Let $T \in \mathcal{D}'(\mathbb{R}^d)$ be a distribution. For $h > 0$ and $1 \leq j \leq d$ define $\tau_{h,i} : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$ by $(\tau_h \varphi)(x) = \varphi(x - he_i)$.

a) Justify why $(\tau_{h,i}T)(\varphi) := T(\tau_{-h,i}\varphi)$ defines a distribution.

b) Show that

$$\lim_{h\to 0} \frac{T - \tau_{h,i}T}{h} = D^{e_i}T \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

In this sense, the distributional derivatives are still limits of difference quotients.

Ex 8.2 (Fourier transform and distributional derivatives*)
Let $\alpha \in \mathbb{N}_0^d$ be a multi-index.

a) Show that the function $\mathbb{R}^d \to \mathbb{R} : \ x \mapsto x^\alpha$ is a tempered distribution.

b) Prove that for any $T \in \mathscr{S}'(\mathbb{R}^d)$

$$\widehat{D^\alpha T} = (ik)^\alpha\, \widehat{T} \quad \text{and} \quad \widehat{x^\alpha T} = (iD)^\alpha \widehat{T}.$$

Hint: Use the corresponding identities for the Fourier transform on $\mathscr{S}(\mathbb{R}^d)$.

c) Show that $\widehat{\delta_0} = 1$ and $\hat{1} = (2\pi)^d\delta_0$. Then demonstrate the following identities in $\mathscr{S}'(\mathbb{R}^d)$:

$$\widehat{D^\alpha\delta_0} = (ik)^\alpha \quad \text{and} \quad \widehat{x^\alpha} = (2\pi)^d(iD)^\alpha\delta_0.$$

Ex 8.3 (Fourier transform of p.v.$(1/x)$**)**
Let

$$T = \text{p.v.}\Big(\frac{1}{x}\Big).$$

a) Show that T is a tempered distribution on $\mathbb{R}$.

Hint: Use the formula for T derived in the solution to Ex. 7.1.

b) Show that $\widehat{T}$ is a solution of the differential equation in $\mathscr{S}'(\mathbb{R})$

$$iD\widehat{T} = 2\pi\delta_0.$$

Hint: Start with the identity $x \cdot T = 1$.

c) Use b) to compute that $\widehat{T} = -i\pi\,\text{sign}$, where sign is the signum function

$$\text{sign}(x) = 1 \quad (x > 0), \quad \text{sign}(x) = -1 \quad (x < 0), \quad \text{sign}(x) = 0 \quad (x = 0).$$

Hint: Employ Ex 7.3 and the fact that T is an **odd distribution**, i.e., $T(\varphi(-\cdot)) = -T(\varphi)$.

Ex 8.4 (Two applications of the fundamental lemma of the calculus of variations)
a) For an open set $\Omega \subset \mathbb{R}^d$, we define the so-called **Sobolev space**

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) : \, D^{e_i} f \in L^p(\Omega) \; \forall i = 1, \dots, d\},$$

where D^{e_i} denotes the distributional derivative. In this case we say that f is **weakly differentiable** and $D^{e_i} f$ is the weak i-th partial derivative. Show that the weak derivative is unique.

b) Let $\Omega \subset \mathbb{R}^d$ be open and $1 \leq p < +\infty$. Show that $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.
Hint: Recall that by the Riesz representation theorem, the dual space of $L^p(\Omega)$ can be identified with $L^q(\Omega)$ with $p^{-1} + q^{-1} = 1$.